to use the partial differential and integral techniques, each in its appropriate region, to solve problems that could be intractable by either one individually.

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Short Papers

A Method for Computing Edge Capacitance of Finite and Semi-Infinite Microstrip Lines

T. ITOH, R. MITTRA, AND R. D. WARD

***Abstract*—This short paper describes a method for computing the edge capacitance of finite or semi-infinite sections of microstrip transmission lines. The approach is based on Galerkin's method applied in the Fourier-transform domain. It is mathematically simple and requires the inversion of rather small-size matrices.**

Introduction

In this short paper, a new method is developed for calculating the fringe (excess) capacitance due to an abrupt truncation of a uniform microstrip line. In contrast to the conventional matrix formulation in the space domain, the method to be presented here is based upon an application of Galerkin's method in the spectral or Fourier-transform domain. The spectral-domain approach has been successfully applied to a number of other problems [1]–[3]. It is particularly suitable for handling open-region problems of the type considered in this short paper.

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Formulation and Method of Solution

In the TEM approximation, it is assumed that the discontinuity capacitance may be computed from the knowledge of the field solution derived in the static limit. This is done by first solving Poisson's equation for the potential function ϕ for the geometry under consideration. (The geometry is shown in Fig. 1.) This, in turn, requires the solution of the equation

$$\nabla^2\phi(x, y, z) = -\frac{1}{\epsilon_0}\rho(x, z)\delta(y)$$
$$\rho(x, z) = 0, \qquad |x| > W/2, \qquad |z| > l/2 \tag{1}$$

where ϵ_0 is the free-space permittivity, $\delta(y)$ is the delta function, and $\rho(x, z)$ is the charge distribution on the strip. The strip is assumed to have infinitesimal thickness and to be perfectly conducting. The ground plane and the dielectric substrate are also assumed to be lossless. Next, we introduce the two-dimensional Fourier transform of the potential ϕ via

$$\tilde{\phi}(\alpha, y, \beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x, y, z)\exp j(\alpha x + \beta z)\, dxdz. \tag{2}$$

Taking the transform of (1), we obtain

$$\left[\frac{\partial^2}{\partial y^2} - (\alpha^2 + \beta^2)\right]\tilde{\phi}(\alpha, y, \beta) = -\frac{1}{\epsilon_0}\tilde{\rho}(\alpha, \beta)\delta(y) \tag{3}$$

where $\tilde{\rho}$ is the transform of charge distribution defined by

$$\tilde{\ }(\alpha, \beta) = \int_{-l/2}^{l/2}\int_{-W/2}^{W/2}\rho(x, z)\exp j(\alpha x + \beta z)\, dxdz. \tag{4}$$

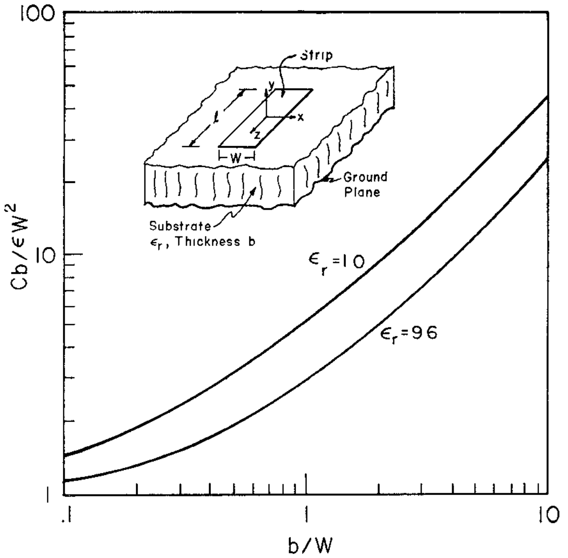

Fig. 1. Capacitance of a square section of microstrip.

We assume that the potentials of the strip and the ground plane are maintained at 1 and 0 V, respectively.

The boundary and continuity conditions are written as

1) $\tilde{\phi}(\alpha, -b, \beta) = 0$
2) $\tilde{\phi}(\alpha, y, \beta) \to 0 \quad \text{as } y \to +\infty$
3) $\tilde{\phi}(\alpha, 0+, \beta) = \tilde{\phi}(\alpha, 0-, \beta)$
4) $\frac{\partial \tilde{\phi}}{\partial y}(\alpha, 0+, \beta) - \epsilon_r \frac{\partial \tilde{\phi}}{\partial y}(\alpha, 0-, \beta) = -\frac{1}{\epsilon_o}\tilde{\rho}(\alpha, \beta).$

The appropriate form of the solution of (3) is

$$\tilde{\phi}(\alpha, y, \beta) = \begin{cases} A(\alpha, \beta) \sinh \sqrt{\alpha^2 + \beta^2}\, y, & -b < y < 0 \\ B(\alpha, \beta) \exp \{-\sqrt{\alpha^2 + \beta^2}\, y\}, & y > 0 \end{cases} \tag{5}$$

where $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are unknown coefficients. Note that the choice of the representation in (5) automatically satisfies conditions 1) and 2). Next we substitute (5) into conditions 3) and 4), and eliminate $A(\alpha, \beta)$ and $B(\alpha, \beta)$ to express the solution for the potential on the strip in the form

$$G(\alpha, \beta)\tilde{\rho}(\alpha, \beta) = \tilde{\phi}_i(\alpha, 0, \beta) + \tilde{\phi}_o(\alpha, 0, \beta) \tag{6}$$

where

$$\tilde{\phi}_i = \iint_{\text{on strip}} \phi(x, 0, z) \exp j(\alpha x + \beta z)\, dx dz = \frac{4}{\alpha\beta} \sin \frac{\alpha W}{2} \sin \frac{\beta l}{2} \tag{7a}$$

$$\tilde{\phi}_o = \iint_{\text{outside of strip}} \phi(x, 0, z) \exp j(\alpha x + \beta z)\, dx dz \tag{7b}$$

$$G(\alpha, \beta) = \frac{1}{\epsilon_o \sqrt{\alpha^2 + \beta^2}\, [1 + \epsilon_r \coth \sqrt{\alpha^2 + \beta^2}\, b]}. \tag{7c}$$

Note that G is the transform of Green's function, and that the algebraic product in the left-hand side of (6) corresponds to the surface convolution integral appearing in the space-domain analysis. This feature is very important and useful in the actual numerical calculation because the computation of the surface convolution integral is a time-consuming operation.

As a first step toward solving (6), we expand the unknown charge distribution in terms of a set of known basis functions

$$\tilde{\rho}(\alpha, \beta) = \sum_{n=1}^{N} d_n \tilde{\zeta}_n(\alpha, \beta). \tag{8}$$

It may be noted that the basis functions in the transform domain $\tilde{\zeta}_n$ are the Fourier transforms of functions ζ_n that have a finite support in the space domain.

The next step for deriving a matrix equation for the unknown coefficients d_n is to take the inner product of (6) with one of the basis functions $\tilde{\phi}_n$. This gives

$$\sum_{n=1}^{N} K_{mn} d_n = f_m, \qquad m = 1, 2, \cdots, N \tag{9}$$

where

$$K_{mn} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\zeta}_m(\alpha, \beta) G(\alpha, \beta) \tilde{\zeta}_n(\alpha, \beta)\, d\alpha d\beta \tag{10}$$

$$f_m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\zeta}_m(\alpha, \beta) \tilde{\phi}_i(\alpha, 0, \beta)\, d\alpha d\beta = \left(\frac{1}{2\pi}\right)^2 \int_{-l/2}^{l/2}\int_{-W/2}^{W/2} \zeta_m(x, z)\, dx dz. \tag{11}$$

The unknown potential $\tilde{\phi}_o$ was eliminated using Parseval's relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\zeta}_m(\alpha, \beta) \tilde{\phi}_o(\alpha, 0, \beta)\, d\alpha d\beta = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta_m(x, z) \cdot \text{FT}^{-1}[\tilde{\phi}_o]\, dx dz = 0$$

because the inverse transforms FT^{-1} of $\tilde{\zeta}_m$ and $\tilde{\phi}_o$ are nonzero only in complementary regions. Once (9) is solved for d_n, the total capacitance for the strip may be computed from the expression

$$C = \int_{-l/2}^{l/2}\int_{-W/2}^{W/2} \rho(x, z)\, dx dz = (2\pi)^2 \sum_{n=1}^{N} d_n f_n. \tag{12}$$

It should be pointed out that the capacitance obtained from the use of (12) is always smaller than the correct value. This follows from the fact that Galerkin's method is equivalent to the variational approximation.

Numerical Procedure and Results

The choice of the basis functions is rather arbitrary as long as they satisfy the required condition that they are zero in the appropriate range. Experience with two-dimensional problems has shown that the polynominals of $|x|$ are very suitable for the uniform line case [1], [2]. This prompts us to choose the following functions for the three-dimensional problem at hand:

$$\zeta_n(x, z) = \begin{cases} |x|^{k-1} |z|^{j-1}, & \text{on the strip} \\ \quad n = 1(k = 1, j = 1), n = 2(k = 2, j = 1), \cdots \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

The transforms of the basis functions may be obtained by using (2). The accuracy of the results improves with the use of more than one basis function, although the one-term approximation is used here. Numerical experiments show, however, that the use of two terms improves the accuracy only slightly, and the increase in computational labor does not usually justify this effort. It will be seen shortly that the calculations based on the one-term approximation compare very favorably with those reported elsewhere using other techniques.

Fig. 1 shows the computed values of capacitance for a square section of microstrip line ($l = W$) for dielectric constants of 1 and 9.6. The plotted values of capacitance are normalized with respect to $\epsilon W^2/b$, which is the value of the capacitance of a parallel plate structure with $\epsilon = \epsilon_r \epsilon_o$. As expected, the normalized capacitance approaches unity as b/W becomes small. The results in Fig. 1 are indistinguishable from those obtained by Farrar and Adams [4], who used the point-matching technique, and Reitan [5] who employed the method of subareas for $\epsilon_r = 1.0$.

The fringing capacitance at the end of the open-circuited microstrip line may be defined as

$$C_{ex} = \lim_{l \to \infty} \tfrac{1}{2}[C(l) - lC_o] \tag{14}$$

where $C(l)$ is the total capacitance of the section of length l and width W, C_o is the line capacitance per unit length of a uniform line of the same width, and the factor $\frac{1}{2}$ accounts for the discontinuities at both ends of the strip. It should be noted that C_{ex} is not variational, although the expression for $C(l)$ given earlier is stationary. In the calculation, l is not infinite, but some finitely large value beyond which the change of $C(l) - lC_o$ is negligible.

The fringe capacitance is sometimes expressed in terms of a hypothetical extension of the microstrip line by a small amount δl

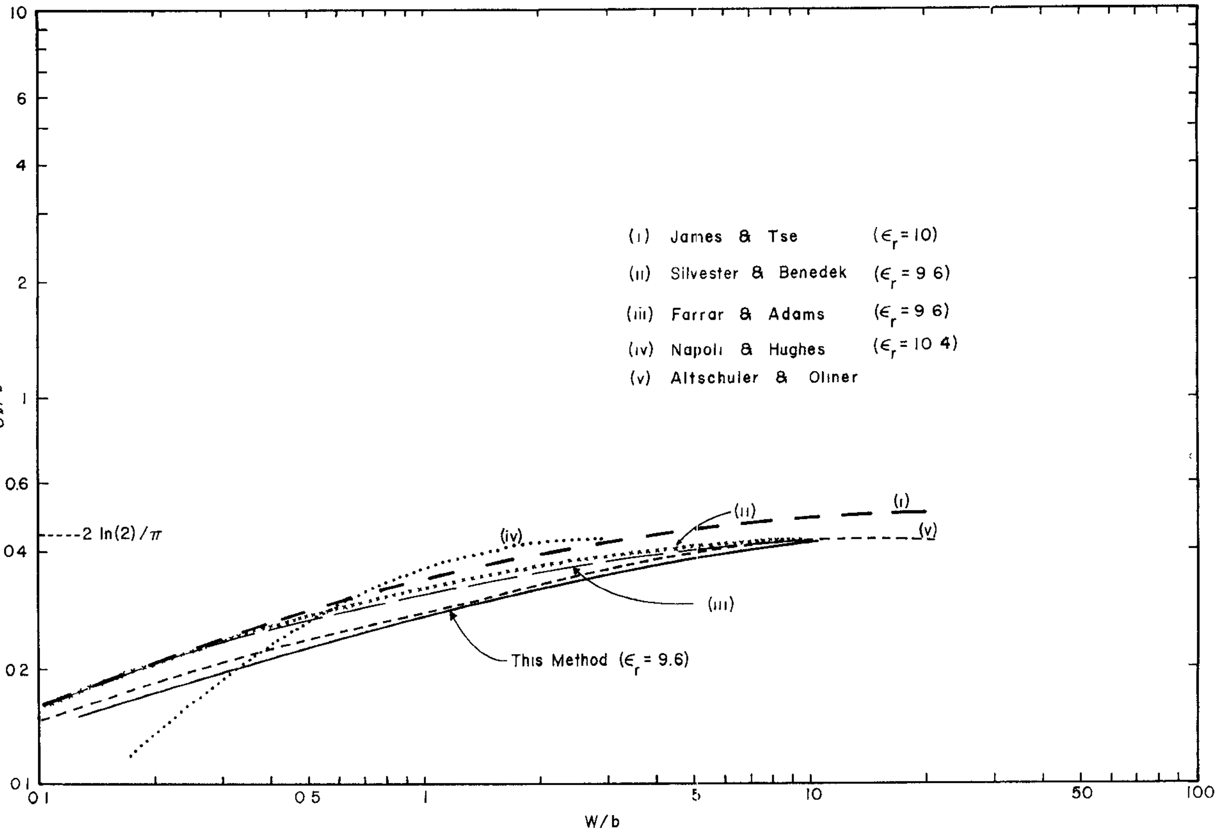

Fig. 2. End effect of a semi-infinite microstrip line.

[6], [7]. This quantity was calculated by using the expressions for C_{ex} given earlier, and the results obtained are shown in Fig. 2. The same quantity has been computed by James and Tse [7] by using an alternative approach, and their results are also included in Fig. 2 for ease of comparison. It is evident that the numerical results obtained in this short paper compare favorably with Farrar and Adams, as well as others.

It may be useful to quote some typical computation time for calculating $C(l)$ by (12). Typical time of the CDC G-20 computer was about 60 s for this calculation (execution time). The above computer is approximately ten times slower than the IBM 360/75. To minimize the computation time for C_{ex} given in (14), the choice of l is important. A numerical experiment shows that, if $l \gtrsim 10$ W, $C(l)$ increases linearly with l. Hence the limiting process can be omitted for this choice of l. Furthermore, since the computation of C_o requires less than 5 s (execution time), the computation time of C_{ex} is also about 60 s.

In conclusion, the method described in this short paper has many advantages, one of which is its numerical efficiency. Another feature is that it is quite general, since many other types of junctions and finite structures can be solved by the present method, either in its present form or with some modifications. Some examples of such structures are gaps in the uniform strip, T junction, etc., that are currently under investigation.

Finally, it should be mentioned that Maeda [8] has recently reported a method for analyzing the gap structure in the microstrip line. The approach outlined in this short paper is believed to be numerically more efficient, since the expression for Green's function in the transform domain is a closed form in contrast to a slowly converging series in the space domain.

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A Quasi-Dynamic Method of Solution of a Class of Waveguide Discontinuity Problems

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***Abstract*—It is shown that, if expansion terms of *all* the modes appearing in the Green's function for the problem are retained, the singular integral equation method can be made to apply by generating a differential equation for this integral. The solution of the differential equation is straightforward, and the inversion of the resulting integral equation then follows standard methods. The process is applied in detail to the case of the capacitive diaphragm, and the results compared to the quasi-static method with correction terms. The results are close for small guide widths, but the present method should give superior results if the guide width permits some overmoding.**

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